# Disc-Ring Network: A Novel Architecture of Interconnection Network

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***Abstract*—In this paper, we will introduce a novel family of interconnection network topologies, named disc-ring networks. Disc-ring networks possess many desirable topological properties in building parallel machines, such as fixed degree, small diameter, hamiltonian decomposable, etc. We will study some topological properties of disc-ring networks. Furthermore, we also present an efficient routing algorithm for disc-ring networks.**

***Keywords*— Interconnection networks, disc-ring network, hypercube, hamiltonian decomposable, diameter**

## 1. INTRODUCTION

Parallel computing is important for speeding up computation. The design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [1, 2, 3, 5, 7, 9, 10, 11], and the desirable properties of an interconnection network include symmetry, small diameter, relatively small degree, embedding capabilities, scalability, fault-tolerant robustness, efficient routing, and hamiltonian decomposition. Among those proposed interconnection networks, the hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [13] In the literature, many variants of hypercube, including twisted cube [9], locally twisted cube [14], crossed cube [6, 7], augmented cube [2], and Möbius cube [3], etc, have been studied. These variants possess better topology properties than hypercube. For example, the diameter of these variants of hypercube is about half of the network diameter of the comparable hypercube. The architecture of an interconnection network is usually modeled by a graph in which the nodes represent the processing elements and the edges represent the communication links. In this paper, we will use graph and network, vertex and node, and edge and link interchangeably.

In this paper, we will introduce a simple and regular architecture called *disc-ring network*. Disc-ring network is constructed from two rings with the same number of nodes by adding some additional links between the two rings. We will analyze the topological properties of disc-ring networks. We derive the upper bound on the diameter of a disc-ring network and show that they are hamiltonian decomposable, i.e., the edges of a disc-ring network can be partitioned into disjoint Hamiltonian cycles. We then present and analyze an efficient routing algorithm for disc-ring networks.

The rest of the paper is organized as follows. In Section 2, we introduce some notations used throughout the paper. We then define a family of interconnection networks, namely disc-ring networks. Section 3 examines the topology properties of disc-ring networks. In Section 4, we give an efficient routing algorithm in disc-ring networks and analyze its performance. Finally, we conclude this paper in Section 5.

## 2. DISC-RING NETWORK

We usually use a graph to represent the topology of an interconnection network. A graph $G = (V, E)$ is a pair of the node set $V$ and the edge set $E$, where $V$ is a finite set and $E$ is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We will use $V(G)$ and $E(G)$ to denote the node set and the edge set of $G$, respectively. If $(u, v)$ is an edge in a graph $G$, we say that $u$ is *adjacent* to $v$ and $u$, $v$ are *incident* to edge $(u, v)$. A *neighbor* of a node $v$ in a graph $G$ is any node that is adjacent to $v$. Moreover, we use $N_G(v)$ to denote the set of neighbors of $v$ in $G$. The subscript '$G$' of $N_G(v)$ can be removed from the notation if it has no ambiguity. A graph $G$ is called $k$-regular if $|N_G(v)| = k$ for any node $v$ in $G$. The distance, denoted by $dist(u, v)$, between node $u$ and node $v$ in a graph is the length of a shortest path from $u$ to $v$. The *diameter*, denoted by $diam(G)$, of a graph $G$ is the maximum distance between any two nodes in it.

Let $G = (V, E)$ be a graph with node set $V$ and edge set $E$. A (simple) path $P$ of length $l$ in $G$, denoted by $v_0 \to v_1 \to \dots \to v_{l-1} \to v_l$, is a sequence $(v_0, v_1, \dots, v_{l-1}, v_l)$ of nodes such that $(v_i, v_{i+1}) \in E$ for $0 \le i \le l-1$. The first node $v_0$ and the last node $v_l$ visited by $P$ are denoted by $start(P)$ and $end(P)$, respectively, and they are called the *end nodes* of $P$. In addition, $P$ is a cycle if $|V(P)| \ge 3$ and $end(P)$ is adjacent to $start(P)$. A path $P = v_0 \to v_1 \to \dots \to v_{l-1} \to v_l$ may contain another subpath $Q$, denoted as $v_0 \to v_1 \to \dots \to v_i \to Q \to v_j \to \dots \to v_{l-1} \to v_l$, where $Q = v_i \to v_{i+1} \to \dots \to v_j$ for $0 \le i \le j \le l$. A path (or cycle) in $G$ is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every node of $G$ exactly once. Two paths (or cycles) $P_1$ and $P_2$ connecting a node $u$ to a node $v$ are said to be *edge-disjoint* if and only if $E(P_1) \cap E(P_2) = \phi$. Two paths (or cycles) $Q_1$ and $Q_2$ of graph $G$ are called *node-disjoint* if and only if $V(P_1) \cap V(P_2) = \phi$. Two node-disjoint paths $Q_1$ and $Q_2$ can be *concatenated* into a path, denoted by $Q_1 \Rightarrow Q_2$, if $end(Q_1)$ is adjacent to $start(Q_2)$.

For positive integers $z$ and $m$, define $z_{||m}$ to be $z$ if $z \ge 0$, and to be $z+m$ otherwise. And, $z_{\%m}$ denotes the remainder of the division of $z$ by $m$.

We define a novel interconnection network, namely *disc-ring network*, as follows. A disc-ring network, represented by $D(m, d)$, consists of two rings, where $1 \le d \le m$ and each ring contains m nodes. One ring is called *inner ring* and the other is called *outer ring*. Each node is labeled by a sequence $z_1z_2$ of two integers $z_1$, $z_2$, where $z_1$ is called the first index while $z_2$ is called the second index. If node $z_1z_2$ is in outer ring, then $z_1 = 0$; otherwise, $z_1 = 1$. The nodes in each ring is labeled from 0 to $m-1$ sequentially in the counterclockwise manner; that is, for node $z_1z_2$ in one ring, $0 \le z_2 \le m-1$. The links between two nodes in the same ring are defined as follows: Each node $z_1z_2$ is adjacent to node $z_1x$ for $x \in \{(z_2+1)_{\%m}, (z_2-1)_{||m}\}$. The links between two nodes in the distinct rings are defined as follows: For any node $0z_2$ in the outer ring, there is a link connecting it to node $1y$ for $y \in \{z_2, (z_2+1)_{\%m}, (z_2+2)_{\%m}, \dots, (z_2+d-1)_{\%m}\}$. The formally definition of disc-ring network is introduced as follows.

**Definition 1.** The disc-ring network, represented by $D(m, d)$, consists of outer and inner rings, where each ring contains $m$ nodes and $1 \le d \le m$. The node set of $D(m, d)$ is $\{z_1z_2 \mid z_1 = 0 \text{ or } 1$, and $z_2 \in \{0, 1, \dots, m-1\}\}$, where $z_1z_2$ is a sequence of two integers and is a label of a node. Node $0z_2$ is in the outer ring while node $1z_2$ is in the inner ring. For any node $z_1z_2$, there is a link connecting it to node $z_1x$ for $x = (z_2+1)_{\%m}$. For any node $0z_2$, $0 \le z_2 \le m-1$, there is a link connecting it to node $1y$ for $y \in \{z_2, (z_2+1)_{\%m}, (z_2+2)_{\%m}, \dots, (z_2+d-1)_{\%m}\}$.

By the above definition, the disc-ring network $D(m, d)$ contains $2m$ nodes and is a $(d+2)$-regular and undirected graph. Thus, $D(m, d)$ contains $m(d+2)$ links (edges). For example, Fig. 1 depicts $D(6, 3)$ and $D(5, 2)$. We can easily verify that $D(m, 1)$ is a prism graph (or called circular ladder graph) [4, 8] and $D(4, 1)$ is isomorphic to a 3-dimensional hypercube.

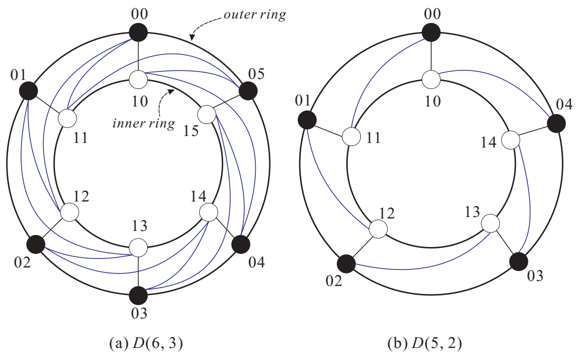


Fig. 1. The disc-ring networks $D(6, 3)$ and $D(5, 2)$, where the black circles denote the outer ring nodes while those white circles denote the inner ring nodes.

It is not difficult to verify from the definition of disc-ring network that the following lemma holds true.

**Lemma 1.** *Let $D(m, d)$ be a disc-ring network. Then, the following statements hold true:*
*(1) Each node $0z_2$, $0 \le z_2 \le m-1$, of $D(m, d)$ is in the outer ring and is adjacent to node $1y$ in the inner ring for $y \in \{z_2, (z_2+1)_{\%m}, (z_2+2)_{\%m}, \dots, (z_2+d-1)_{\%m}\}$.*
*(2) Each node $1z_2$, $0 \le z_2 \le m-1$, of $D(m, d)$ is in the inner ring and is adjacent to node $0z$ in the outer ring for $z \in \{z_2, (z_2-1)_{||m}, (z_2-2)_{||m}, \dots, (z_2-d+1)_{||m}\}$.*

For example, node 05 in $D(6, 3)$ shown in Fig. 1(a) is adjacent to nodes 15, 10, 11; and node 10 is adjacent to nodes 00, 05, 04.

## 3. Topology Properties of Disc-Ring Networks

In this section, we will examine the major topology properties of disc-ring networks, such as diameter and hamiltonian decomposition.

We first observe that the disc-ring network is outer-inner ring symmetric. If we exchange the nodes of outer ring and inner ring, and relabel the nodes of each ring sequentially in a clockwise manner, the resultant network is isomorphic to the original disc-ring network. For example, relabeling $D(6, 3)$ shown in Fig. 1(a) will obtain a network shown in Fig. 2 which is isomorphic to $D(6, 3)$. We then have the following lemma.

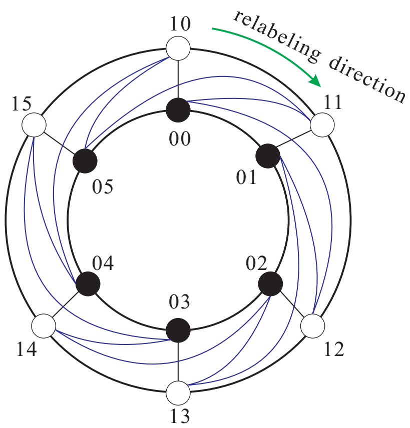


Fig. 2. Exchanging and relabeling the nodes of outer ring and inner ring for $D(6, 3)$ shown in Fig. 1(a).

**Lemma 2.** *Let $D(m, d)$ be a disc-ring network. Then, exchanging the outer ring and inner ring of $D(m, d)$ and relabeling the nodes of each ring in the clockwise manner will result in an isomorphic network for $D(m, d)$.*

Next, we will examine the diameter of disc-ring network $D(m, d)$. We first compute the diameter of a ring as follows.

**Lemma 3.** *The diameter of a ring with m nodes is* $\lfloor m/2 \rfloor$.

*Proof:* Let $R$ be a ring with $m$ nodes. We first label the nodes of $R$ from 0 to $m$–1 sequentially in the counterclockwise manner. Since the symmetric structure of a ring, we can pick the node 0 and compute the shortest path from node 0 to any other node in $R$. Since each node $x$, $0 \le x \le m-1$, of $R$ is adjacent to nodes $(x+1)_{\%m}$ and $(x-1)_{\|m}$, the distance between node 0 and node $z$ is $\min\{z-0, m-z\}$, where $z-0$ is the length of path from node 0 to node $z$ in the counterclockwise direction and $m-z$ is the length of path from node 0 to node $z$ in the clockwise direction. When $z = \lfloor m/2 \rfloor$, the distance between node 0 and node $z$ is the largest. Therefore, the diameter of a ring with $m$ nodes is $\lfloor m/2 \rfloor$. □

For determining the diameter of disc-ring network $D(m, d)$, we first consider the case of $d \le 2$. The following lemma can be found in [12] and can be verified by the similar arguments in Lemma 3.

**Lemma 4.** *[12] The diameter of disc-ring network $D(m, d)$ with $d \ge 2$ is* $\lfloor (m+3-d)/2 \rfloor$.

For example, the diameter of $D(5, 2)$ equals to the diameter of a ring with 6 nodes, where the bold lines in Fig. 3 depict such a ring (00 → 11 → 12 → 13 → 14 → 10) with diameter being equal to that of $D(5, 2)$.

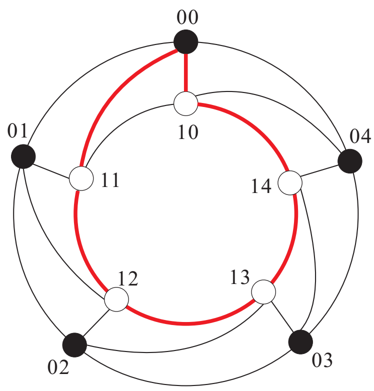


Fig. 3. The ring with diameter being equal to diameter of $D(5, 2)$, where the bold lines indicate such a ring embedded in $D(5, 2)$.

We then consider the case of $d \ge 3$ for computing an upper bound on the diameter of disc-ring network $D(m, d)$. We will obtain an upper bound on the diameter of $D(m, d)$ with $d \ge 3$ to be $\lfloor m/(d-1) \rfloor + 2$ in the following lemma.

**Lemma 5.** *An upper bound on the diameter of disc-ring network $D(m, d)$ with $d \ge 3$ is* $\lfloor m/(d-1) \rfloor + 2$.

*Proof:* By Lemma 2 and the symmetric structure of ring, we can pick the node 00 to compute an upper bound of the shortest path from node 00 to any other node in $D(m, d)$. We first partition the node set of $D(m, d)$ into four subsets as follows:

$V_A = \{1a \mid 0 \le a \le \lfloor m/2 \rfloor\}$,

$V_B = \{1b \mid \lfloor m/2 \rfloor + 1 \le b \le m-1\}$,
$V_C = \{0c \mid 1 \le c \le \lfloor m/2 \rfloor\}$, and
$V_D = \{0d \mid \lfloor m/2 \rfloor + 1 \le d \le m-1\}$.

The above partition is shown in Fig. 4.

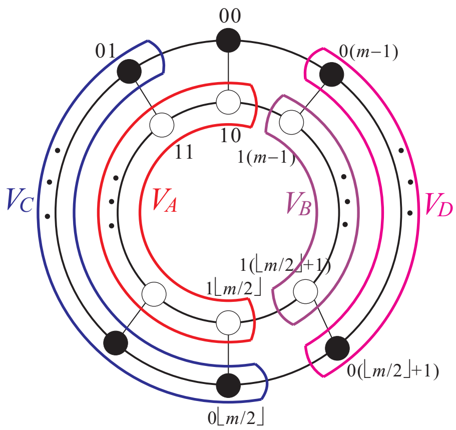


Fig. 4. The partition of node set in $D(m, d)$ with $d \ge 3$ for computing an upper bound on the diameter of $D(m, d)$.

We will compute the paths from node 00 to nodes of $V_A$, $V_B$ $V_C$, $V_D$ such that their lengths are not greater than $\lfloor m/(d-1) \rfloor + 2$. Then, an upper bound of diameter of $D(m, d)$ is obtained.

We first consider the nodes of $V_A$. Let $1z_a$ be a node of $V_A$, where $0 \le z_a \le \lfloor m/2 \rfloor$. By Lemma 1, $1z_a$ is adjacent to nodes $0z$ for $z \in \{z_a, (z_a-1)_{\|m}, (z_a-2)_{\|m}, \ldots, (z_a-d+1)_{\|m}\}$ which are in the outer ring. We construct a path $P_a$ from node 00 to node $1z_a$ as follows. Initially, let $P_a$ = 00. Repeat the following steps until $end(P_a) = 1z_a$. If node $1z_a$ is a adjacent to node $end(P_a)$, then let $P_a = P_a \rightarrow 1z_a$ and stop. Otherwise, if $end(P_a)$ is in the inner ring, then let $end(P_a) = 1z$ and let $P_a = P_a \rightarrow 0z$; else let $end(P_a) = 0z$, $z' = z+d-1$, and let $P_a = P_a \rightarrow 1z'$. Fig. 5 reveals the construction of path $P_a$. For example, given node $z_a$ = 13 in $D(6, 3)$, the above procedure constructs path $P_a$ = 00 → 12 → 13. We now analyze the length of $P_a$. By Lemma 1, if $0 \le z_a \le d-1$, then the length of $P_a$ is 1. Assume that $z_a > d-1$ and $k(d-1)+1 \le z_a \le (k+1)(d-1)$. By the above procedure, the length of $P_a$ is not greater than $2k+1$. By the definition of $V_A$, $k(d-1) < \lfloor m/2 \rfloor$ and hence, $2k+1 < \lfloor m/(d-1) \rfloor + 1$, i.e., $2k+1 \le \lfloor m/(d-1) \rfloor$. Thus, $dist(00, 1z_a) \le \lfloor m/(d-1) \rfloor$.

We next consider the nodes of $V_D$. Let $0z_d$ be a node of $V_D$, where $\lfloor m/2 \rfloor + 1 \le z_d \le m-1$. By similar construction for $P_a$, we can obtain a path $P_d$ starting from 00 and ending at $0z_d$ as follows. Initially, let $P_d$ = 00 → 10. Repeat the following steps until $end(P_d) = 0z_d$. If $0z_d$ is a adjacent to node $end(P_d)$, then let $P_d = P_d \rightarrow 0z_d$ and stop. Otherwise, if node $end(P_d)$ is in the outer ring, then let $end(P_d) = 0z$ and let $P_d = P_d \rightarrow 1z$; else let $end(P_d) = 1z$, $z' = (z-(d-1))_{\|m}$, and let $P_d = P_d \rightarrow 0z'$. Fig. 5 also depicts such a construction of path $P_d$. For example, given node $z_d$ = 04 in $D(6, 3)$, the above procedure constructs path $P_d$ = 00 → 10 → 04. We now analyze the length of $P_d$. By Lemma 1, if $m-(d-1) \le z_d \le m-1$, then the length of $P_d$ is 2. Assume that $z_d < m-(d-1)$ and $m-j(d-1)+1 \le z_d \le m-(j+1)(d-1)$. By the above construction of $P_d$, the length of $P_d$ is not greater than $2(j+1)$. By the definition of $V_D$, $m-j(d-1) > \lfloor m/2 \rfloor + 1$ and hence, $\lfloor m/(d-1) \rfloor + 2 > 2(j+1)$, i.e., $\lfloor m/(d-1) \rfloor + 1 \ge 2(j+1)$. Thus, $dist(00, 0z_d) \le \lfloor m/(d-1) \rfloor + 1$.

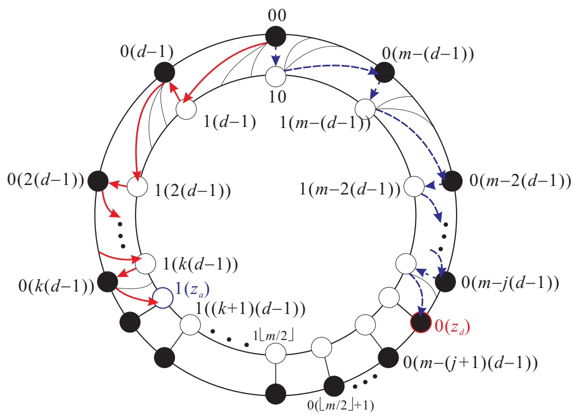


Fig. 5. The routing paths $P_a$ and $P_d$ from node 00 to nodes of $V_A$ and $V_D$, respectively, in $D(m, d)$ with $d \ge 3$, where solid arrow lines indicate $P_a$ and dashed arrowed lines indicate $P_d$.

For the nodes in $V_B$ and $V_C$, let $1z_b \in V_B$ and let $0z_c \in V_C$. To obtain a path $P_b$ from 00 to $1z_b$, we first compute a path $P'_d$ between 00 and $0z_b$, where $0z_b \in V_D$, and then extend it to visit node $1z_b$. That is, $P_b = P'_d \rightarrow 1z_b$. By the above analysis, the length of $P'_d$ is not greater than $\lfloor m/(d-1) \rfloor + 1$. Thus, the length of $P_b$ is not greater than $\lfloor m/(d-1) \rfloor + 2$. To obtain a path $P_c$ from 00 to $0z_c$, we first compute a path $P'_a$ between 00 and $1z_c$, where $1z_c \in V_A$, and then extend it to visit node $0z_c$. That is, $P_c = P'_a \rightarrow 0z_c$. By the previous analysis, the length of $P'_a$ is not greater than $\lfloor m/(d-1) \rfloor$. Thus, the length of $P_c$ is not greater than $\lfloor m/(d-1) \rfloor + 1$.

It follows from the above arguments, we have that the distance between node 00 and any other node in $D(m, d)$ is not greater than $\lfloor m/(d-1) \rfloor + 2$.

Thus, the diameter of $D(m, d)$ is not greater than $\lfloor m/(d-1) \rfloor + 2$. This completes the proof. □

It follows from Lemmas 4 and 5 that the following theorem holds true.

**Theorem 6.** *An upper bound on the diameter of disc-ring network* $D(m, d)$ *is* $\lfloor m/(d-1) \rfloor + 2$ *if* $d \le 2$; *and is* $\lfloor m/(d-1) \rfloor + 2$ *otherwise. That is,*

$$diam(D(m,d)) \le \begin{cases} \lfloor (m+3-d)/2 \rfloor & \text{, if } d \le 2; \\ \lfloor m/(d-1) \rfloor + 2 & \text{, if } d \ge 3. \end{cases}$$

Next, we will study another topology property of disc-ring networks that they contain a *hamiltonian decomposition*. We first give the definition of hamiltonian decomposition as follows.

**Definition 2.** A *hamiltonian decomposition* of a network $G$ is a partition of its edge set into Hamiltonian cycles; i.e., the edges of $G$ can be partitioned into disjoint Hamiltonian cycles.

By the above definition, a hamiltonian decomposition of a $k$-regular network consists of $\lfloor k/2 \rfloor$ edge-disjoint Hamiltonian cycles. Thus, if $D(m, d)$ contains a hamiltonian decomposition then it contains $\lfloor (d+2)/2 \rfloor$ edge-disjoint Hamiltonian cycles. The following theorem shows that $D(m, d)$ contains a hamiltonian decomposition.

**Theorem 7.** *The disc-ring network* $D(m, d)$ *admits a hamiltonian decomposition.*

*Proof:* Obviously, $D(m, 1)$ contains a Hamiltonian cycle that traverses the outer ring and then the inner ring. In the following, we will assume that $d$ is even. We will prove by induction on $d$ that $D(m, d)$ contains $\lfloor (d+2)/2 \rfloor$ edge-disjoint Hamiltonian cycles. Initially, let $d = 2$. Then, $D(m, 2)$ contains two edge-disjoint Hamiltonian cycles $P_1$ and $P_2$. The construction of $P_1$ and $P_2$ is as follows. Let $P_1^o = 00 \to 01 \to 02 \to \ldots \to 0(m-2) \to 0(m-1)$ and let $P_1^i = 1(m-1) \to 1(m-2) \to \ldots \to 11 \to 10$. That is, $P_1^o$ traverses all nodes of the outer ring from node 00 in the clockwise direction, and $P_1^i$ traverses all nodes of the inner ring from node 10 in the counterclockwise direction. Let $P_1 = P_1^o \Rightarrow P_1^i$. In addition, let $P_2 = 00 \to 11 \to 01 \to 12 \to \ldots \to 0j \to 1(j+1) \to \ldots \to 0(m-2) \to 1(m-1) \to 10 \to 0(m-1)$; that is, $P_2$ alternately traverses the nodes of outer ring and inner ring. For example, $P_1$ and $P_2$ of $D(5, 2)$ are shown in Fig. 6. Then, $P_1$ and $P_2$ form two edge-disjoint Hamiltonian cycles of $D(m, 2)$. Thus, the lemma holds true when $d = 2$.

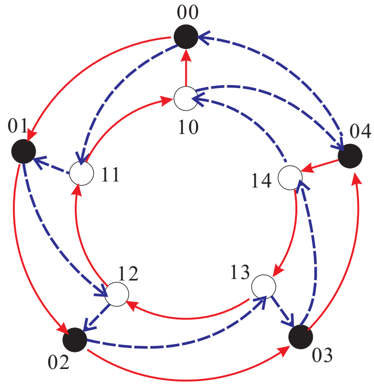


Fig. 6. Two edge-disjoint Hamiltonian cycles of $D(5, 2)$, where solid arrow lines indicate one Hamiltonian cycle and dashed arrow lines indicate the other cycle.

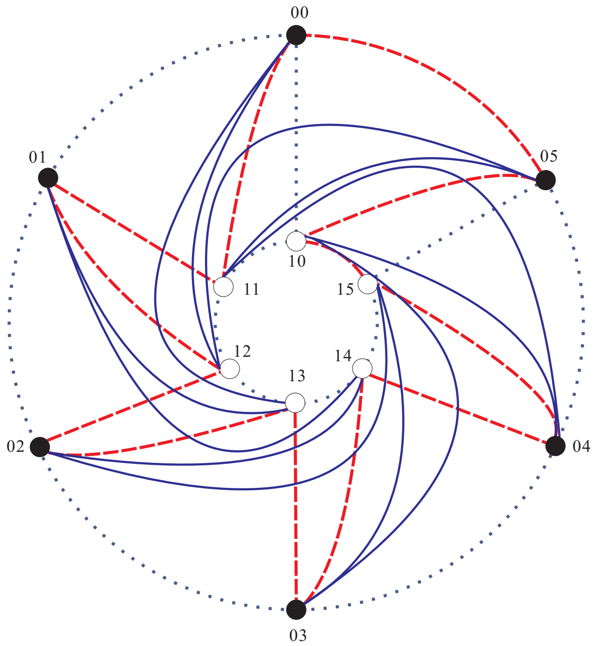


Fig. 7. Three edge-disjoint Hamiltonian cycles for disc-ring network $D(6, 4)$, where dashed lines indicate a Hamiltonian cycle, dotted lines depict another Hamiltonian cycle, and solid lines indicate the other Hamiltonian cycle.

Assume that $d = k$, $k \ge 2$, $D(m, k)$ contains $\lfloor (k+2)/2 \rfloor$ edge-disjoint Hamiltonian cycles. Let $P$ be the set of such edge-disjoint Hamiltonian cycles. Now, consider that $d = k+2$. Let $P_o^* = \{00, 0(m-1), 0(m-2), \ldots, 03, 02, 01\}$ and let $P_i^* = \{1(k+1), 1(k), 1(k-1), \ldots, 1(k+4), 1(k+3), 1(k+2)\}$. That is, $P_o^*$ is a set of nodes in outer ring in the counterclockwise manner, and $P_i^*$ is a set of nodes in inner ring in the counterclockwise manner. We then alternately traverse the nodes of $P_o^*$ and $P_i^*$ to obtain a cycle $P^* = 00 \to 1(k+1) \to$

$0(m-1) \to 1k \to 0(m-2) \to 1(k-1) \to \ldots \to 03 \to 1(k+4) \to 02 \to 1(k+3) \to 01 \to 1(k+2)$. Then, $P \cup \{P^*\}$ forms a set of $\lfloor \frac{k+4}{2} \rfloor$ edge-disjoint Hamiltonian cycles of $D(m, k+2)$. For example, Fig. 7 shows three edge-disjoint Hamiltonian cycles of $D(6, 4)$. Thus, the lemma holds true when $d = k+2$. By induction, $D(m, d)$ contains $\lfloor \frac{d+2}{2} \rfloor$ edge-disjoint Hamiltonian cycles, and, hence, the lemma holds true. □

## 4. Routing in Disc-Ring Networks

For efficiency of communication, a simple and fast routing algorithm should ensure that a message can be forwarded from a source node to a destination node along a path. In this section, we will propose an efficient routing algorithm for disc-ring network $D(m, d)$. Because the routing path in $D(m, 1)$ and $D(m, 2)$ can be easily constructed, we will assume that a disc-ring network $D(m, d)$ with $d \geq 3$ is given. Our routing algorithm is based on the proof of Lemma 5. The algorithm is given by a pair of source and destination nodes and constructs a routing path $P$ from the source node to the destination node such that the length of $P$ is not greater than $\lfloor \frac{m}{d-1} \rfloor + 2$. That is, the algorithm achieves the upper bound of diameter of $D(m, d)$ which is given in Lemma 5. The basic idea is as follows. Let $V_A$, $V_B$, $V_C$, $V_D$ be the partition of node set in the proof of Lemma 5, where Fig. 4 depicts the partition. By Lemma 2, we can exchange the outer ring and inner ring to obtain an isomorphic disc-ring network. The exchanging resultant network is shown in Fig. 8. The exchanging resultant disc-ring network preserves the topology of the original disc-ring network. Fig. 8 also depicts the partition of node set with respect to node 00.

Observe the topology of the exchanging resultant disc-ring network, we have that the construction of routing path will be the same no matter whether the source node is located in the outer ring or not. The routing algorithm is formally presented as follows.

**Algorithm** Routing-DiscRing
**Input:** The source node $xy$ and the destination node $x'y'$ in disc-ring network $D(m, d)$ with $d \geq 3$.
**Output:** A routing path $P$ from node $xy$ to node $x'y'$.
**Method:**

1. Initially, let $P = xy$;
2. **if** $x = 0$ **then**

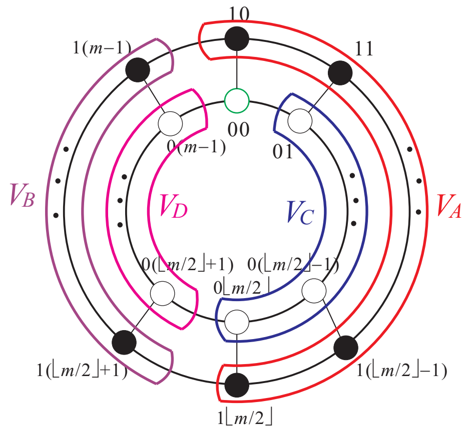


Fig. 8. The resultant network of $D(m, d)$ by exchanging the outer ring and inner ring and relabeling the nodes in the clockwise manner.

3. let $V_A = \{1a \mid a = y, (y+1)_{\%m}, (y+2)_{\%m}, \ldots, (y+\lfloor \frac{m}{2} \rfloor)_{\%m}\}$,
   $V_B = \{1b \mid b = (y+\lfloor \frac{m}{2} \rfloor+1)_{\%m}, (y+\lfloor \frac{m}{2} \rfloor+2)_{\%m}, \ldots, (y+m-1)_{\%m}\}$,
   $V_C = \{0c \mid c = (y+1)_{\%m}, (y+2)_{\%m}, \ldots, (y+\lfloor \frac{m}{2} \rfloor)_{\%m}\}$, and let
   $V_D = \{0d \mid d = (y+\lfloor \frac{m}{2} \rfloor+1)_{\%m}, (y+\lfloor \frac{m}{2} \rfloor+2)_{\%m}, \ldots, (y+m-1)_{\%m}\}$;
4. **else**
5. let $V_A = \{0a \mid a = (y+\lfloor \frac{m}{2} \rfloor)_{\%m}, (y+\lfloor \frac{m}{2} \rfloor+1)_{\%m}, \ldots, (y+m)_{\%m}\}$,
   $V_B = \{0b \mid b = (y+1)_{\%m}, (y+2)_{\%m}, \ldots, (y+\lfloor \frac{m}{2} \rfloor-1)_{\%m}\}$,
   $V_C = \{1c \mid c = (y+\lfloor \frac{m}{2} \rfloor)_{\%m}, (y+\lfloor \frac{m}{2} \rfloor+1)_{\%m}, \ldots, (y+m-1)_{\%m}\}$, and let
   $V_D = \{1d \mid d = y, (y+1)_{\%m}, (y+2)_{\%m}, \ldots, (y+\lfloor \frac{m}{2} \rfloor-1)_{\%m}\}$;
6. **if** $x'y' \in V_A$ or $x'y' \in V_C$ **then**
7. **repeat**
8. **if** $x'y' \in N(end(P))$ **then** let $P = P \to x'y'$;
9. **else**
10. let $end(P) = x^*y^*$;
11. **if** $x \neq x^*$ **then** let $P = P \to xy^*$;
12. **else** let $P = P \to (1-x)z$, where $z = (y^*+(d-1))_{\%m}$ if $x = 0$; and $z = (y^*-(d-1))_{\|m}$ otherwise;
13. **until** $end(P) = x'y'$;
14. **else** // $x'y' \in V_B$ or $x'y' \in V_D$
15. **repeat**

16. **if** $x'y' \in N(end(P))$ **then** let $P = P \rightarrow x'y'$;
17. **else**
18. let $end(P) = x^*y^*$;
19. **if** $x = x^*$ **then** let $P = P \rightarrow (1-x)y^*$;
20. **else** let $P = P \rightarrow xz$, where $z = (y^*-(d-1))_{||m}$ if $x = 0$; and $z = (y^*+(d-1))_{\%m}$ otherwise;
21. **until** $end(P) = x'y'$;
22. **output** "$P$" as a routing path starting from $xy$ and ending at $x'y'$, and **terminate**.

For example, given source node 04 and destination node 11 in disc-ring network $D(6, 3)$ shown in Fig. 1(a), the routing algorithm will produce the routing path $P = 04 \rightarrow 10 \rightarrow 11$.

Now, we analyze the complexity of Algorithm Routing-DiscRing. Given source node $xy$ and destination node $x'y'$, deciding whether $x'y' \in V_A$ can be computed in $O(1)$ time by comparing the indices of the input nodes. Thus, lines 2–5 of Algorithm Routing-DiscRing can be done in $O(1)$ time. By Lemma 5, Lines 7–13 and Lines 15–21 are repeated by at most $O(\lfloor m/(d-1) \rfloor)$ times. Since every line in the repeated process can be done in $O(1)$ time, Algorithm Routing-DiscRing runs in $O(\lfloor m/(d-1) \rfloor)$ time. We then have the following theorem.

**Theorem 8.** *Given a source node s and a destination node t in disc-ring network D(m, d) with* $d \geq 3$, *Algorithm Routing-DiscRing computes a routing path P starting from s and ending at t in* $O(\lfloor m/(d-1) \rfloor)$ *time, where P achieves the upper bound* $\lfloor m/(d-1) \rfloor + 2$ *on the diameter of* $D(m, d)$.

## 5. Concluding Remarks

In this paper, we first introduce disc-ring networks. The structure of disc-ring networks is simple and easy to implement. We then examine two topology properties on them. We provide an upper bound on diameter of disc-ring network and show that disc-ring network admits a hamiltonian decomposition. We also present an efficient routing algorithm on disc-ring network. In the future, we will use disc-ring network as infrastructure architecture to construct a novel compound network constructed from disc-ring network and hypercube or its variants.


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